Example 1: Rectangular Three Point Bending

A rectangular specimen is subjected to a three-point bending test. The specimen is 10 centimeters long, 10 millimeters wide (b) and 10 millimeters tall (h). The specimen is placed on two supports that are 5 cm apart (L), and the actuator is applying a force in the exact middle of the two supports (L/2). Immediately before failure, the Instron records a force (F) of 50N, and a deformation (δ) of 2mm. We need to determine the maximum flexural strength (σ), and Young’s Modulus (E) of the specimen.

To accomplish this task, we are going to use the two following equations:

\[ \sigma = \frac{My}{I} \quad \text{and} \quad E = \frac{mL^3}{4bh^3} \]  

(Eq. 1.1 and 1.2)

Where \( M \) is the moment (or torque) applied at the middle of the specimen, \( y \) is the distance from the center of the specimen to the convex surface, \( I \) is the “polar moment of inertia,” a term used to define
how the geometry of the specimen influences its reaction to loads, and $m$ is the slope of the linear portion of the force displacement curve.

First, we must calculate the reaction forces at the supports. We have two unknown values, and therefore must use two equations to solve the system. Based on static mechanics, we can use the following two equations:

\[
\sum F_y = 0 \tag{Eq. 1.3}
\]

and

\[
\sum M_{\text{Any Point}} = 0 \tag{Eq. 1.4}
\]

In our case, these equations are as follows:

\[
\sum F_y = -F + F_{\text{Left Support}} + F_{\text{Right Support}} = 0
\]

\[
\sum F_y = -50 + F_{\text{Left Support}} + F_{\text{Right Support}} = 0
\]

Or

\[
F_{\text{Left Support}} + F_{\text{Right Support}} = 50 \tag{Eq. 1.5}
\]

Using the (Eq. 1.4), we find:

\[
\sum M_{\text{Left Support}} = (-F \times \frac{L}{2}) + (F_{\text{Right Support}} \times L) = 0
\]

\[
(-50 \times \frac{0.05}{2}) + (F_{\text{Right Support}} \times 0.05) = 0
\]

Solving for $F_{\text{Right Support}}$ we find:

\[
F_{\text{Right Support}} = \frac{(50 - 0.05 \times 2)}{0.05} = 25N
\]

Substituting the value of 25N for $F_{\text{Right Support}}$ back into (Eq. 1.5), we find:

\[
F_{\text{Left Support}} + 25 = 50
\]

Therefore

\[
F_{\text{Left Support}} = 25N
\]

Now that we have solved for the reaction forces at the supports, we can calculate the moment acting at the midpoint of the specimen by looking at half of the specimen and using the following equation:
\[
\text{Moment} = \sum \text{Force} \cdot \text{distance} \quad \text{(Eq. 1.6)}
\]

...in our case...

\[
M = F_{\text{Right Support}} \cdot \frac{L}{2}
\]

\[
M = 25 \times 0.025 = 0.625Nm
\]

Next we calculate \( y \), the distance from the center of the specimen to the convex surface:

\[
y = \frac{h}{2} \quad \text{(Eq. 1.7)}
\]

...in our case...

\[
y = \frac{0.010}{2} = 0.005m
\]

Finally, we must calculate \( I \), the polar moment of inertia, for our rectangular cross-section. The equation for a solid rectangular cross-section is:

\[
I_{\text{rectangle}} = \frac{bh^3}{12} \quad \text{(Eq. 1.8)}
\]

If we plug in our values, we get:

\[
I_{\text{rectangle}} = \frac{0.01 \times 0.01^3}{12} = 8.33e^{-10}
\]

Now that we have calculated \( M, y \), and \( I \), we are ready calculate the maximum flexural stress of the specimen, by using (Eq. 1.1):

\[
\sigma = \frac{My}{I} = \frac{0.625 \times 0.005}{8.33e^{-10}} = 3.75e^6 \text{Pa or } 3.75 \text{ MPa}
\]

Finally, we can calculate the Young’s modulus of the material by plugging in our values to (Eq. 1.2).

\[
E = \frac{mL^3}{4bh^3}
\]

If we assume that our test was linear until failure, we can calculate the slope of the force displacement curve to be the failure load (50N) divided by the failure displacement (2mm):

\[
E = \frac{50 \times 0.05^3}{0.002 \times 4 \times 0.01 \times 0.01^3} = 78.125 \text{ MPa}
\]